

# Action principle for the connection dynamics of scalar-tensor theories

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A first-order action for scalar-tensor theories of gravity is proposed. The Hamiltonian analysis of the action gives a connection dynamical formalism, which is equivalent to the connection dynamics derived from the geometrical dynamics by canonical transformations. Therefore, the action principle underlying loop quantum scalar-tensor theories is recovered.

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## I. INTRODUCTION

Modified gravity theories have recently received increased attention in issues related to the "dark Universe" and nontrivial tests on gravity beyond general relativity (GR). Since 1998, a series of independent astronomic observations implied that our Universe is currently undergoing a period of accelerated expansion [1]. This caused the "dark energy" problem in the framework of GR. It is thus reasonable to consider the possibility that GR is not a valid theory of gravity on a galactic or cosmological scale. A simple and typical modification of GR is the so-called  $f(R)$  theory of gravity [2]. Besides  $f(R)$  theories, a well-known competing relativistic theory of gravity was proposed by Brans and Dicke in 1961 [3], which is apparently compatible with Mach's principle. To represent a varying "gravitational constant", a scalar field is nonminimally coupled to the metric in Brans-Dicke theory. To be compared with the observational results within the framework of broad class of theories, the Brans-Dicke theory was generalized by Bergmann [4] and Wagoner [5] to general scalar-tensor theories (STT). Scalar-tensor modifications of GR are also popular in unification schemes such as string theory (see, e.g., [6] [7] [8]). Note that the metric  $f(R)$  theories and Palatini  $f(R)$  theories are equivalent to the special kinds of STT with the coupling parameter  $\omega = 0$  and  $\omega = -\frac{3}{2}$  respectively [2], while the original Brans-Dicke theory is the particular case of constant  $\omega$  and vanishing potential of  $\phi$ .

In the past two decades, a nonperturbative quantization of GR, called loop quantum gravity (LQG), has matured [9] [10] [11] [12]. It is remarkable that both  $f(R)$  theories and STT can be nonperturbatively quantized by extending the LQG techniques [13] [14] [15]. Thus LQG is extended to more general metric theories of gravity [16]. The background independent quantization method relies on the key observations that these theories can be cast into the connection dynamical formulations with the structure group  $SU(2)$ . The connection dynamical formulation of  $f(R)$  theories and STT were obtained by canonical transformations from their geometrical dynamics [13] [14] [15]. However, the action principle for the connection dynamics of either  $f(R)$  theories or STT is still lacking. The purpose of this paper is to fill out this gap. We

will propose a first-order action for general STT of gravity, which includes  $f(R)$  theories as special cases. The connection dynamical formalism will be derived from this action by Hamiltonian analysis. It turns out that this connection dynamics is equivalent to that derived from the geometrical dynamics by canonical transformations. Hence, loop quantum STT, as well as loop quantum  $f(R)$  theories, have got their foundation of action principle.

Throughout the paper, we use the Greek alphabet for space-time indices, capital Latin alphabet  $I, J, K, \dots$ , for internal Lorentzian indices, Latin alphabet  $a, b, c, \dots$ , for spatial indices, and  $i, j, k, \dots$ , for internal  $SU(2)$  indices. The other convention are as follows. The internal Minkowski metric is denoted by  $\eta_{IJ} = \text{diag}(-1, 1, 1, 1)$ . The Hodge dual of a differential form  $F_{IJ}$  is denoted by  $\star F_{IJ} = \frac{1}{2}\epsilon_{IJKL}F^{KL}$ , where  $\epsilon_{IJKL}$  is the internal Levi-Civita symbol. The antisymmetry of a tensor  $A_{IJ}$  is defined by  $A_{[IJ]} = A_{IJ} - A_{JI}$ .

## II. EQUATIONS OF MOTION

Let us consider the following first-order action:

$$S[e, \omega, \phi] = \int \left[ \frac{1}{2} \phi e e_I^\alpha e_J^\beta (R_{\alpha\beta}{}^{IJ} + \frac{1}{\gamma} \star R_{\alpha\beta}{}^{IJ}) - \frac{1}{2} K(\phi) e e^\mu e_I^\nu (\partial_\mu \phi) \partial_\nu \phi - e V(\phi) \right] d^4x, \quad (1)$$

where  $e = \det(e_\mu^I)$  is the determinant of the right-handed tetrad  $e_\mu^I$ ,  $R_{\alpha\beta}{}^{IJ} = \partial_{[\alpha} \omega_{\beta]}^{IJ} + \omega_{[\alpha}^{IK} \omega_{\beta]K}{}^J$  is the curvature of the  $SL(2, \mathbb{C})$  spin connection  $\omega_\alpha^{IJ}$ ,  $V(\phi)$  is the potential of the scalar field  $\phi$ , where  $\phi$  satisfies  $\phi > 0$ ,  $K$  is an arbitrary function of  $\phi$ , and  $\gamma$  is an arbitrary real number. Note that a first-order action for gravity non-minimally coupled with a scalar field was proposed in [17], which will lead to geometrical dynamics rather than connection dynamics.

Now let us calculate the equations of motion of action (1). It is convenient to employ the notation  $\overset{\gamma}{U}_{IJ} = U_{IJ} + \frac{1}{\gamma} \star U_{IJ}$ . One can derive the useful relations

$$(U_{[I|K|} V_{J]}^K) = \overset{\gamma}{U}_{[I|K|} V_{J]}^K = U_{[I}^K \overset{\gamma}{V}_{J]K},$$

$$\overset{\gamma}{U}_{IJ} V^{IJ} = U^{IJ} \overset{\gamma}{V}_{IJ}.$$

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The variation of action (1) with respect to  $\omega_\alpha^{IJ}$  gives

$$\mathcal{D}_\alpha(\phi e e_I^{[\alpha} e_J^{\beta]}) + \frac{1}{\gamma} \mathcal{D}_\alpha(\phi e \epsilon_{IJ}^{KL} e_K^\alpha e_L^\beta) = 0. \quad (2)$$

Here the operator  $\mathcal{D}_\alpha$  is defined as

$$\mathcal{D}_\alpha e_\beta^I = \partial_\alpha e_\beta^I - \Gamma_{\alpha\beta}^\gamma e_\gamma^I + \omega_\alpha^{IJ} e_{\beta J}, \quad (3)$$

where  $\Gamma_{\alpha\beta}^\gamma$  is a torsion-free affine connection. From the relation (2) we have (see [19] for details)

$$\mathcal{D}_{[\alpha}(\sqrt{\phi} e_{\beta]}^I) = 0, \quad (4)$$

which tells us that the spin connection  $\omega_\alpha^{IJ}$  are compatible with tetrad  $\sqrt{\phi} e_\alpha^I$ . On the other hand, the variation of action (1) with respect to the tetrad  $e_\mu^I$  gives

$$\begin{aligned} & \phi(e_{\beta I} e_J^\mu \tilde{R}_{\alpha\mu}^{IJ} - \frac{1}{2} g_{\alpha\beta} e_I^\mu e_J^\nu \tilde{R}_{\mu\nu}^{IJ}) \\ &= K((\partial_\alpha \phi) \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} (\partial_\mu \phi) \partial^\mu \phi) - g_{\alpha\beta} V, \end{aligned} \quad (5)$$

where we have used

$$\delta e_\mu^I = -e_\mu^J e_\alpha^I \delta e_J^\alpha, \quad (6)$$

$$\delta e_I^\mu = -e_I^\alpha e_J^\mu \delta e_\alpha^J, \quad (7)$$

$$\delta e = e e_I^\mu \delta e_\mu^I, \quad (8)$$

$$\delta(e e^{I\alpha} e^{J\beta}) = e(e^{I\alpha} e^{J\beta} e_K^\sigma - e^{I\sigma} e^{J\beta} e_K^\alpha - e^{I\alpha} e^{J\sigma} e_K^\beta) \delta e_\sigma^K. \quad (9)$$

Taking account of Eq.(4), Eq.(5) becomes

$$\phi \tilde{G}_{\alpha\beta} = K((\partial_\alpha \phi) \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} (\partial_\mu \phi) \partial^\mu \phi) - g_{\alpha\beta} V, \quad (10)$$

where  $\tilde{G}_{\alpha\beta}$  is the Einstein tensor of  $\sqrt{\phi} e_\alpha^I$ . Using the identity

$$\begin{aligned} \phi \tilde{G}_{\alpha\beta} &= \phi G_{\alpha\beta} + \frac{3}{2\phi} ((\partial_\alpha \phi) \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} (\partial_\mu \phi) \partial^\mu \phi) \\ &\quad - \nabla_\alpha \nabla_\beta \phi + g_{\alpha\beta} \nabla_\mu \nabla^\mu \phi, \end{aligned} \quad (11)$$

where  $G_{\alpha\beta}$  is the Einstein tensor of  $e_\alpha^I$ , Eq.(10) becomes

$$\begin{aligned} \phi G_{\alpha\beta} &= (K - \frac{3}{2\phi}) ((\partial_\alpha \phi) \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} (\partial_\mu \phi) \partial^\mu \phi) \\ &\quad + \nabla_\alpha \nabla_\beta \phi - g_{\alpha\beta} \nabla_\mu \nabla^\mu \phi - g_{\alpha\beta} V. \end{aligned} \quad (12)$$

Finally, the variation of action (1) with respect to the scalar field  $\phi$  gives

$$\tilde{R} + 2K \nabla_\mu \nabla^\mu \phi + K' (\partial_\mu \phi) \partial^\mu \phi - 2V' = 0, \quad (13)$$

where a prime over a function represents a derivative with respect to the argument  $\phi$ . We define a new function

$$\frac{\omega(\phi)}{\phi} := K(\phi) - \frac{3}{2\phi}. \quad (14)$$

Then it is straightforward to transform Eq.(12) and (13) into the form in [15]. Hence the first-order action (1) gives exactly the equations of motion of STT.

### III. HAMILTONIAN ANALYSIS

Let the spacetime  $M$  be topologically  $\Sigma \times \mathbb{R}$  for some 3-manifold  $\Sigma$ . One introduces a foliation of  $M$  and a time-evolution vector field  $t^a$  in it. In a coordinate system adapted to the (3+1)-decomposition of spacetime, the Lagrangian in action (1) can be written as

$$\begin{aligned} \mathcal{L} &= \phi e e_I^t e_J^\gamma \tilde{R}_{ta}^{IJ} + \frac{1}{2} \phi e e_I^a e_J^\gamma \tilde{R}_{ab}^{IJ} \\ &\quad - \frac{1}{2} K(\phi) e e^{tI} e_I^t \dot{\phi}^2 - K(\phi) e e^{tI} e_I^a \dot{\phi} \partial_a \phi \\ &\quad - \frac{1}{2} K(\phi) e e^{aI} e_I^b (\partial_a \phi) \partial_b \phi - e V(\phi), \end{aligned} \quad (15)$$

where a dot over a letter represents a derivative with respect to the time coordinate  $t$ . By Legendre transformation, the momentum conjugate to the configuration variables  $\omega_a^{IJ}$  and  $\phi$  are defined respectively as

$$\pi_{IJ}^a := 2 \frac{\delta \mathcal{L}}{\delta \omega_a^{IJ}} = \phi e e_{[I}^t e_{J]}^a, \quad (16)$$

$$\pi := \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = -K(\phi) e e^{tI} e_I^t \dot{\phi} - K(\phi) e e^{tI} e_I^a \partial_a \phi. \quad (17)$$

It is obvious that the momentum  $\pi_{IJ}^a$  are subject to 6 constraints:

$$\Phi^{ab} := \frac{1}{2\phi^2} \pi^{aIJ} \pi_{IJ}^b \approx 0. \quad (18)$$

Let  $N^I$  be a normalized timelike vector in internal space such that  $\eta_{IJ} N^I N^J = -1$ . Then we can split the tetrad as

$$e_{0I} = N N_I + N^a V_{aI}, \quad e_{aI} = V_{aI}, \quad N^I V_{aI} = 0, \quad (19)$$

where  $V_a^I$  plays the role of co-triad,  $N$  and  $N^a$  are respectively the lapse function and shift vector. The corresponding Hamiltonian on spatial slice  $\Sigma$  can be derived as a linear combination of constraints as

$$H_{tot} = \int_\Sigma d^3x \left( -\frac{1}{2} \omega_t^{IJ} \mathcal{G}_{IJ} + N^a C_a + \frac{N^2}{e} C + \lambda_{ab} \Phi^{ab} \right), \quad (20)$$

where  $\lambda_{ab}$  is a Lagrange multiplier, and the Gaussian, diffeomorphism and Hamiltonian constraints read, respectively,

$$\mathcal{G}_{IJ} = \mathcal{D}_a \pi_{IJ}^a \approx 0, \quad (21)$$

$$C_a = \frac{1}{2} \pi_{IJ}^b \tilde{R}_{ab}^{IJ} + \pi \partial_a \phi \approx 0, \quad (22)$$

$$\begin{aligned} C &= \frac{1}{2\phi} \pi_I^a \pi_{JK}^b \tilde{R}_{ab}^{IJ} - \frac{K(\phi)}{4\phi^2} \pi^{aIJ} \pi_{IJ}^b \partial_a \phi \partial_b \phi \\ &\quad + \frac{\pi^2}{2K(\phi)} + \frac{V}{\phi^3} \sqrt{\det(-\frac{1}{2} \pi_{IJ}^a \pi^{bIJ})} \approx 0. \end{aligned} \quad (23)$$

Here we have defined

$$\mathcal{D}_a \pi_{IJ}^a := \partial_a \pi_{IJ}^a + \omega_{aI}^K \pi_{KJ}^a - \omega_{aJ}^K \pi_{KI}^a \quad (24)$$

and used the identity

$$e = \frac{N^2}{\phi^3 e} \sqrt{\det(-\frac{1}{2}\pi_{IJ}^a \pi^{bIJ})}. \quad (25)$$

The fundamental Poisson brackets read

$$\{\gamma \omega_a^{IJ}(x), \pi_{KL}^b(y)\} = \delta_a^b \delta_K^{[I} \delta_L^{J]} \delta^3(x-y), \quad (26)$$

$$\{\phi(x), \pi(y)\} = \delta^3(x-y). \quad (27)$$

It is easy to check that, the constraints  $\mathcal{G}^{IJ}$  are the generators of internal gauge transformations, and the combination

$$\begin{aligned} \mathcal{H}_a &:= C_a - \frac{1}{2} \omega_a^{IJ} \mathcal{G}_{IJ} \\ &= \frac{1}{2} (\gamma \pi_{IJ}^b \partial_a \omega_b^{IJ} - \partial_b (\gamma \pi_{IJ}^b \omega_a^{IJ})) + \pi \partial_a \phi \end{aligned} \quad (28)$$

are the generators of spatial diffeomorphisms. The constraints algebra can be calculated as

$$\{\frac{1}{2} \mathcal{G}^{IJ} [\Lambda_{IJ}], \frac{1}{2} \mathcal{G}^{KL} [\Omega_{KL}]\} = \mathcal{G}^{IJ} [\Lambda_I^K \Omega_{KJ}], \quad (29)$$

$$\{\frac{1}{2} \mathcal{G}^{IJ} [\Lambda_{IJ}], \mathcal{H}_a [N^a]\} = -\frac{1}{2} \mathcal{G}^{IJ} [N^a \partial_a \Lambda_{IJ}], \quad (30)$$

$$\{\frac{1}{2} \mathcal{G}^{IJ} [\Lambda_{IJ}], C[N]\} = 0, \quad (31)$$

$$\{\frac{1}{2} \mathcal{G}^{IJ} [\Lambda_{IJ}], \Phi^{ab} [\kappa_{ab}]\} = 0, \quad (32)$$

$$\begin{aligned} \{\omega_{aIJ}(x), \mathcal{H}_b [N^b]\} &= N^b \partial_b \omega_{aIJ} + (\partial_a N^b) \omega_{bIJ} \\ &= \mathcal{L}_{N^b} \omega_{aIJ}(x), \end{aligned} \quad (33)$$

$$\begin{aligned} \{\pi^{aIJ}(x), \mathcal{H}_b [N^b]\} &= N^b \partial_b \pi^{aIJ} - \pi^{bIJ} \partial_b N^a + (\partial_b N^b) \pi^{aIJ} \\ &= \mathcal{L}_{N^b} \pi^{aIJ}(x), \end{aligned} \quad (34)$$

$$\{\phi(x), \mathcal{H}_b [N^b]\} = N^b \partial_b \phi = \mathcal{L}_{N^b} \phi(x), \quad (35)$$

$$\{\pi(x), \mathcal{H}_b [N^b]\} = N^b \partial_b \pi + \pi \partial_b N^b = \mathcal{L}_{N^b} \pi(x), \quad (36)$$

$$\{\mathcal{H}_a [M^a], \mathcal{H}_b [N^b]\} = \mathcal{H}_a [M^b \partial_b N^a - N^b \partial_b M^a], \quad (37)$$

$$\begin{aligned} \{C[M], C[N]\} &= -\frac{1}{2} \mathcal{H}_a [\frac{1}{\phi^2} (M \partial_b N - N \partial_b M) \pi^{aIJ} \pi_{IJ}^b] \\ &+ \frac{1}{2} \Phi^{ab} [(M \partial_b N - N \partial_b M) \star \pi^{cIJ} \tilde{R}_{cbIJ}], \end{aligned} \quad (38)$$

$$\{\frac{1}{2} \Phi^{ab} [\kappa_{ab}], \frac{1}{2} \Phi^{cd} [\kappa_{cd}]\} = 0, \quad (39)$$

$$\{C[N], \frac{1}{2} \Phi^{ab} [\kappa_{ab}]\} = \frac{1}{2} \Psi^{ab} [\frac{N}{\phi^3} \kappa_{ab}] - \Phi^{ab} [\frac{N\pi}{K(\phi)\phi} \kappa_{ab}], \quad (40)$$

where  $\mathcal{L}_{N^b}$  denotes the Lie derivative with respect to  $N^b$ , and

$$\Psi^{ab} := \star \pi^{cIJ} \pi_{IK}^{(a} \mathcal{D}_c \pi_J^{b)K} \quad (41)$$

are second-class constraints.

It is not difficult to see that, although there is a non-minimally coupled scalar field, we can still use the same way as in [18] to solve all second-class constraints. Let

$$E_i^a := \pi_{ij}^a, \quad (42)$$

$$\chi_i := -\frac{e_i^t}{e^{t0}}. \quad (43)$$

Then we have

$$E_{[i}^a \chi_{j]} = \pi_{ij}^a. \quad (44)$$

It is convenient to do the gauge fixing

$$\chi_i = 0. \quad (45)$$

Then we get

$$E_i^a = \phi \frac{e}{N} e_i^a, \quad (46)$$

and its conjugate variable

$$A_a^i := \omega_{a0}^i. \quad (47)$$

By solving the second-class constraints, one also obtains

$$\gamma \omega_a^{ij} = \epsilon_k^{ij} [\gamma^{-1} A_a^k - (1 + \gamma^{-2}) \Gamma_a^k], \quad (48)$$

where the  $SU(2)$  spin connection  $\Gamma_a^i$  satisfies

$$D_a E_i^b = \partial_a E_i^b + \Gamma_{ac}^b E_i^c - \Gamma_{ca}^c E_i^b + \epsilon_{ij}^k \Gamma_a^j E_k^b = 0. \quad (49)$$

Here  $\Gamma_{ab}^c$  is the Christoffel connection determined by the spatial metric

$$q^{ab} = E E^{ai} E_i^b, \quad (50)$$

with  $E := 1/\det(E_i^a)$ . The remaining first-class constraints are

$$\mathcal{G}_i = \gamma^{-1} \tilde{\mathcal{D}}_a E_i^a, \quad (51)$$

$$C_a = E_i^b F_{ab}^i + (1 + \gamma^{-2}) (-\gamma A_a^i + \Gamma_a^i) \mathcal{G}_i + \pi \partial_a \phi, \quad (52)$$

$$\begin{aligned} C &= -\gamma^{-1} \frac{1}{2\phi} \epsilon^{ij} E_i^a E_j^b [F_{ab}^k - (\gamma + \gamma^{-1}) R_{ab}^k] \\ &+ \frac{K(\phi)}{2\phi^2} E^{ai} E_i^b (\partial_a \phi) \partial_b \phi + \frac{\pi^2}{2K(\phi)} + \frac{V}{\phi^3} \sqrt{\det(E^{ai} E_i^b)}, \end{aligned} \quad (53)$$

where

$$\tilde{\mathcal{D}}_a E_i^a := \partial_a E_i^a + \gamma \epsilon_{ij}^k A_a^j E_k^a, \quad (54)$$

and  $F_{ab}^i$  and  $R_{ab}^i$  stand for the curvature of  $A_a^i$  and  $\Gamma_a^i$  respectively, i.e.,

$$F_{ab}^i = \partial_{[a} A_{b]}^i + \gamma \epsilon_{jk}^i A_a^j A_b^k, \quad (55)$$

$$R_{ab}^i = \partial_{[a} \Gamma_{b]}^i + \epsilon_{jk}^i \Gamma_a^j \Gamma_b^k. \quad (56)$$

The fundamental Poisson brackets (26) and (27) become

$$\{A_a^i(x), E_j^b(y)\} = \delta_a^b \delta_j^i \delta^3(x-y), \quad (57)$$

$$\{\phi(x), \pi(y)\} = \delta^3(x-y). \quad (58)$$

Therefore, the  $SU(2)$  connection  $A_a^i$  becomes a configuration variable. The final Hamiltonian is a linear combination of first-class constraints as

$$H = \int d^3x (\Lambda^i \mathcal{G}_i + N^a C_a + \frac{N}{3e} C), \quad (59)$$

where  ${}^3e := |\det(e_a^i)|$ . So far we have derived a first-class constraint system by the Hamiltonian analysis of action (1), which gives a connection dynamical formalism of STT of gravity.

It should be noted that the above connection dynamical formalism does not take exactly the same form as that in [15], because of the differences in basic variables. However, the two formulations are equivalent to each other, since they are related by a canonical transformation. To see this, we first define

$$\tilde{K}_a^i := \phi(A_a^i - \gamma^{-1}\Gamma_a^i), \quad (60)$$

$$\tilde{E}_i^a := \phi^{-1}E_i^a. \quad (61)$$

Then we further define

$$\tilde{\pi} := \pi - \frac{1}{\phi}\tilde{K}_a^i\tilde{E}_i^a, \quad (62)$$

$$\tilde{A}_a^i := \tilde{\Gamma}_a^i + \gamma\tilde{K}_a^i, \quad (63)$$

where  $\tilde{\Gamma}_a^i$  is the  $SU(2)$  spin connection satisfying

$$\tilde{D}_a\tilde{E}_i^b = \partial_a\tilde{E}_i^b + \tilde{\Gamma}_{ac}^b\tilde{E}_i^c - \tilde{\Gamma}_{ca}^b\tilde{E}_i^c + \epsilon_{ij}^k\tilde{\Gamma}_a^j\tilde{E}_k^b = 0. \quad (64)$$

Here  $\tilde{\Gamma}_{ab}^c$  is the Christoffel connection determined by the spatial metric

$$\tilde{q}^{ab} = \tilde{E}\tilde{E}^{ai}\tilde{E}^{bi}, \quad (65)$$

with  $\tilde{E} := 1/\det(\tilde{E}_i^a)$ . Using Eqs.(57) and (58), we can get the Poisson brackets between new variables as

$$\{\tilde{A}_a^i(x), \tilde{E}_j^b(y)\} = \gamma\delta_a^b\delta_j^i\delta^3(x-y), \quad (66)$$

$$\{\phi(x), \tilde{\pi}(y)\} = \delta^3(x-y). \quad (67)$$

Taking account of Eq.(14), the constraints (51), (52) and (53) can be written in terms of new variables, up to Gaussian constraint, as

$$\mathcal{G}_i = \gamma(\partial_a\tilde{E}_i^a + \epsilon_{ij}^k\tilde{A}_a^j\tilde{E}_k^a) \approx 0, \quad (68)$$

$$C_a = \gamma^{-1}\tilde{E}_i^b\tilde{F}_{ab}^i + \tilde{\pi}\partial_a\phi \approx 0, \quad (69)$$

$$\begin{aligned} C = & \frac{\phi}{2}\epsilon_i^{lm}\tilde{E}_l^a\tilde{E}_m^b[\tilde{F}_{ab}^i - (\gamma^2 + \frac{1}{\phi^2})\epsilon_{jk}^i\tilde{K}_a^j\tilde{K}_b^k] \\ & + \frac{1}{2\omega(\phi) + 3}\left(\frac{1}{\phi}(\tilde{K}_a^i\tilde{E}_i^a)^2 + 2\tilde{\pi}\tilde{K}_a^i\tilde{E}_i^a + \tilde{\pi}^2\phi\right) \\ & + \frac{\omega(\phi)}{2\phi}\tilde{E}^{ai}\tilde{E}_i^b(\partial_a\phi)\partial_b\phi + \tilde{E}^{ai}\tilde{E}_i^b(\partial_a\partial_b\phi - \tilde{\Gamma}_{ab}^c\partial_c\phi) \\ & + V\sqrt{\det(\tilde{E}^{ai}\tilde{E}_i^a)} \approx 0, \end{aligned} \quad (70)$$

where  $\tilde{F}_{ab}^i$  is defined as

$$\tilde{F}_{ab}^i := \partial_{[a}\tilde{A}_{b]}^i + \epsilon_{jk}^i\tilde{A}_a^j\tilde{A}_b^k. \quad (71)$$

These constraints coincide with those in [15].

## IV. CONCLUDING REMARKS

As candidate modified gravity theories, STT provide the great possibility to account for the dark Universe and some fundamental issues in physics. The nonperturbative loop quantization of STT is based on their connection dynamical formalism obtained in Hamiltonian formulation in [15]. The achievement in this paper is to set up an action principle for the connection dynamics of STT. Since  $f(R)$  theories of gravity can be regarded as the special kinds of STT, our action principle is also valid for the connection dynamics of  $f(R)$  theories. To get the action principle, we first show that the first-order action (1) gives the right equations of motion for general STT. Then a detailed Hamiltonian analysis is done to this action. By a partial gauge fixing, the internal  $SL(2, \mathbb{C})$  group of the theory is reduced to  $SU(2)$ , and the second-class constraints are solved. Thus we obtain a first-class Hamiltonian system with a  $SU(2)$  connection as a configuration variable.

The directly corresponding Hamiltonian connection formulation of action (1) is in Einstein frame, while as shown in [15] the natural connection formulation obtained by canonical transformations in Hamiltonian framework is in Jordan frame. However, they are shown to be equivalent to each other at classical level. Nevertheless, the ambiguity, whether one should start with the Jordan frame or Einstein frame to quantize STT, still exists. Besides providing the action principle for connection dynamics of STT, action (1) also lays the foundation of spinfoam path-integral quantization of STT. We leave this project for future study.

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## Appendix

It is easy to get the Gaussian constraint (68) and diffeomorphism constraint (69) from Eqs.(51) and (52). Now we derive the Hamiltonian constraint (70). From Eqs.(60), (55) and (56), we have

$$F_{ab}^i = \gamma^{-1}R_{ab}^i + D_{[a}\left(\frac{1}{\phi}\tilde{K}_{b]}^i\right) + \frac{\gamma}{\phi^2}\epsilon_{jk}^i\tilde{K}_a^j\tilde{K}_b^k, \quad (72)$$

where

$$D_{[a}\left(\frac{1}{\phi}\tilde{K}_{b]}^i\right) = \partial_{[a}\left(\frac{1}{\phi}\tilde{K}_{b]}^i\right) + \frac{1}{\phi}\epsilon_{jk}^i\Gamma_{[a}^j\tilde{K}_{b]}^k. \quad (73)$$

Hence Eq.(53) becomes

$$\begin{aligned}
C = & \frac{\phi}{2} \epsilon^{ij}_k \tilde{E}_i^a \tilde{E}_j^b \left( R_{ab}^k - \frac{1}{\phi^2} \epsilon^k_{lm} \tilde{K}_a^l \tilde{K}_b^m \right) - \gamma^{-1} \epsilon^{ij}_k \tilde{E}_i^a \tilde{E}_j^b D_a \left( \frac{1}{\phi} \tilde{K}_b^k \right) \\
& + \frac{1}{2\omega(\phi) + 3} \left( \frac{1}{\phi} (\tilde{K}_a^i \tilde{E}_i^a)^2 + 2\tilde{\pi} \tilde{K}_a^i \tilde{E}_i^a + \tilde{\pi}^2 \phi \right) \\
& + \frac{\omega(\phi)}{2\phi} \tilde{E}^{ai} \tilde{E}_i^b (\partial_a \phi) \partial_b \phi + \frac{3}{4\phi} \tilde{E}^{ai} \tilde{E}_i^b (\partial_a \phi) \partial_b \phi \\
& + V \sqrt{\det(\tilde{E}^{ai} \tilde{E}_i^b)}, \quad (74)
\end{aligned}$$

where we have used Eq.(14). On the other hand, we have

$$\Gamma_a^i = \tilde{\Gamma}_a^i + \frac{1}{2\phi} \epsilon^{ik}_j \tilde{E}_a^j \tilde{E}_k^b \partial_b \phi, \quad (75)$$

which implies

$$\begin{aligned}
\epsilon^{ij}_k \tilde{E}_i^a \tilde{E}_j^b R_{ab}^k = & \epsilon^{ij}_k \tilde{E}_i^a \tilde{E}_j^b \tilde{R}_{ab}^k + \frac{1}{\phi} (2\tilde{E}^{ai} \partial_a \tilde{E}_i^c \\
& - \epsilon^{jk}_i \tilde{\Gamma}_a^i \tilde{E}_j^a \tilde{E}_k^c - \tilde{E}_i^a \tilde{E}^{bj} \tilde{E}_j^c \partial_{[a} \tilde{E}_{b]}^i) \partial_c \phi \\
& + \frac{2}{\phi} \tilde{E}^{ai} \tilde{E}_i^c \partial_a \partial_c \phi - \frac{3}{2\phi^2} \tilde{E}^{ai} \tilde{E}_i^c (\partial_a \phi) \partial_c \phi \\
= & \epsilon^{ij}_k \tilde{E}_i^a \tilde{E}_j^b \tilde{R}_{ab}^k - \frac{2}{\phi} \tilde{E}^{ai} \tilde{E}_i^b \tilde{\Gamma}_{ab}^c \partial_c \phi \\
& + \frac{2}{\phi} \tilde{E}^{ai} \tilde{E}_i^c \partial_a \partial_c \phi - \frac{3}{2\phi^2} \tilde{E}^{ai} \tilde{E}_i^c (\partial_a \phi) \partial_c \phi, \quad (76)
\end{aligned}$$

where

$$\tilde{R}_{ab}^i = \partial_{[a} \tilde{\Gamma}_{b]}^i + \epsilon^i_{jk} \tilde{\Gamma}_a^j \tilde{\Gamma}_b^k. \quad (77)$$

Therefore we get

$$\begin{aligned}
C = & \frac{\phi}{2} \epsilon^{ij}_k \tilde{E}_i^a \tilde{E}_j^b \left( \tilde{R}_{ab}^k - \frac{1}{\phi^2} \epsilon^k_{lm} \tilde{K}_a^l \tilde{K}_b^m \right) - \gamma^{-1} \epsilon^{ij}_k \tilde{E}_i^a \tilde{E}_j^b D_a \left( \frac{1}{\phi} \tilde{K}_b^k \right) \\
& + \frac{1}{2\omega(\phi) + 3} \left( \frac{1}{\phi} (\tilde{K}_a^i \tilde{E}_i^a)^2 + 2\tilde{\pi} \tilde{K}_a^i \tilde{E}_i^a + \tilde{\pi}^2 \phi \right) \\
& + \frac{\omega(\phi)}{2\phi} \tilde{E}^{ai} \tilde{E}_i^b (\partial_a \phi) \partial_b \phi + \tilde{E}^{ai} \tilde{E}_i^b (\partial_a \partial_b \phi - \tilde{\Gamma}_{ab}^c \partial_c \phi) \\
& + V \sqrt{\det(\tilde{E}^{ai} \tilde{E}_i^b)}. \quad (78)
\end{aligned}$$

Moreover, since we have

$$\begin{aligned}
\tilde{F}_{ab}^i = & \partial_{[a} \tilde{A}_{b]}^i + \epsilon^i_{jk} \tilde{A}_a^j \tilde{A}_b^k \\
= & \tilde{R}_{ab}^i + \gamma \tilde{D}_{[a} \tilde{K}_{b]}^i + \gamma^2 \epsilon^i_{jk} \tilde{K}_a^j \tilde{K}_b^k, \quad (79)
\end{aligned}$$

Eq.(78) can be written as

$$\begin{aligned}
C = & \frac{\phi}{2} \epsilon^{ij}_k \tilde{E}_i^a \tilde{E}_j^b \left( \tilde{F}_{ab}^k - (\gamma^2 + \frac{1}{\phi^2}) \epsilon^k_{lm} \tilde{K}_a^l \tilde{K}_b^m \right) \\
& - \tilde{E}_i^a (\gamma^{-1} D_a \tilde{G}^i + \gamma \phi \tilde{D}_a \tilde{G}^i) \\
& + \frac{1}{2\omega(\phi) + 3} \left( \frac{1}{\phi} (\tilde{K}_a^i \tilde{E}_i^a)^2 + 2\tilde{\pi} \tilde{K}_a^i \tilde{E}_i^a + \tilde{\pi}^2 \phi \right) \\
& + \frac{\omega(\phi)}{2\phi} \tilde{E}^{ai} \tilde{E}_i^b (\partial_a \phi) \partial_b \phi + \tilde{E}^{ai} \tilde{E}_i^b (\partial_a \partial_b \phi - \tilde{\Gamma}_{ab}^c \partial_c \phi) \\
& + V \sqrt{\det(\tilde{E}^{ai} \tilde{E}_i^b)}. \quad (80)
\end{aligned}$$

Thus modulo Guassian constraint, Eq.(80) coincides with Eq.(70).

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